

26-AMALIY MASHG'ULOT

Ish vaqti va bosqichlari	O'qituvchi faoliyatining mazmuni	Talabalar faoliyatining mazmuni
I-Bosqich O'quv mashg'ulotiga kirish (10 daqiqa)	1.1. Mashg'ulot mavzusi bo'yicha tushuncha berish. 1.2. Mavzuning avvalgi o'tilgan mavzu bilan aloqadorligi bo'yicha tushuncha berish	Tinglaydilar, yozadilar
II-Bosqich Asosiy (65 daqiqa)	2.1. Amaliy mashg'ulot mavzusiga oid nazariy ma'lumot berish. 2.2. Mavzuga oid amaliy topshiriqlar bajarish.	Tinglaydilar, yozadilar, savollar beradi, bajaradilar.
III-Bosqich Yakunlovchi (5 daqiqa)	3.1. Amaliy mashg'ulotni yakunlash 3.2. Talabalarga mustaqil bajarish uchun topshiriqlar berish.	Tinglaydilar, yozadilar, savollar beradi, bajaradilar.

26-MAVZU: O'ZGARUVCHILARNI ALMASHTIRIB INTEGRALLASH

1. Aniqmas integralda o'zgaruvchini almashtirish.

Faraz qilaylik, bizga $I = \int f(x)dx$ integralni hisoblash kerak bo'lsin. Integral ostida shunday $f(x)$ funksiyalar mavjud bo'ladiki, bu funksiyalarning integralini hisoblash uchun yangi o'zgaruvchi kiritishga to'g'ri keladi. Faraz qilaylik, $I = \int f(x)dx$ integralda $x = \varphi(t)$ o'zgaruvchi almashtiraylik, unda $dx = \varphi'(t)dt$ bo'ladi. Ularni integral ostidagi ifodaga qo'ysak, $\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt$ bo'ladi. Bu formula aniqmas integralda o'zgaruvchi almashtirish formulasi deyiladi.

1-misol. $I = \int \frac{dx}{1+\sqrt[3]{x+1}}$ ni hisoblang. Buni hisoblash uchun biz o'zgaruvchi almashtirish usulidan foydalanamiz.

$$x+1=z^3 \text{ desak, } x=z^3-1, \quad dx=3z^2dz$$

$$\begin{aligned}
 I &= \int \frac{dx}{1+\sqrt[3]{x+1}} = \int \frac{3z^2dz}{1+z} = 3 \int \frac{z^2-1+1}{1+z} dz = \\
 &= 3 \left(\int \frac{z^2-1}{1+z} dz + \int \frac{dz}{1+z} \right) = 3 \left(\int \frac{(z-1)(z+2)}{z+1} dz + \int \frac{dz}{1+z} \right) = \\
 &= 3 \left(\frac{z^2}{2} - z + \ln|1+z| + c \right) = -\frac{3^3 \sqrt{(x+1)^2}}{2} - 3^3 \sqrt{x+1} + 3 \ln|1+\sqrt[3]{x+1}| + c
 \end{aligned}$$

2. Aniqmas integralni bo'laklab integrallash.

Bizga differensiallanuvchi bo'lgan $u(x)$ va $v(x)$ funksiyalari berilgan bo'lsin. Bizga ma'lumki, $d(u \cdot v) = vdu + u dv$ edi.

Bu yerdan udv ni topsak, $udv = d(u \cdot v) - vdu$ bo'ladi. Bu tengliklarni integrallasak, $\int udv = \int d(u \cdot v) - \int vdu$, $\int udv = u \cdot v - \int vdu$

Bu formula aniqmas integralda bo'laklab integrallash formulasi deyiladi.

2-misol. $I = \int x \ln x dx$ ni hisoblang.

$$u = \ln x, \quad du = \frac{1}{x} dx, \quad dv = x dx, \quad v = \frac{x^2}{2},$$

$$\begin{aligned} I = \int x \ln x dx &= \frac{\ln x}{2} \cdot x^2 - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{\ln x}{2} \cdot x^2 - \int \frac{x}{2} dx = \\ &= \frac{\ln x}{2} \cdot x^2 - \frac{1}{2} \cdot \frac{x^2}{2} = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + C \end{aligned}$$

Tekshirish. $\int f(x) dx = F(x) + C$

$$\begin{aligned} F'(x) &= \left[\frac{x^2}{2} \cdot \left(\ln x - \frac{1}{2} \right) + C \right]' = \\ &= 2 \cdot \frac{x}{2} \cdot \left(\ln x - \frac{1}{2} \right) + x^2 \cdot \frac{1}{x} = x \ln x - \frac{x}{2} + \frac{x}{2} = x \ln x = f(x) \end{aligned}$$

3- misol. $I = \int \arctg \sqrt{x} dx$ integralni hisoblang.

$u = \arctg \sqrt{x}$ bo'lsa $du = \frac{1}{1+x} \frac{dx}{2\sqrt{x}}$ $dv = dx$ desak $v = x$ bo'ladi. Bo'laklab integrallash formulasiga ko'ra

$$\begin{aligned} I &= \int \arctg \sqrt{x} dx = x \arctg \sqrt{x} - \int \frac{x dx}{2\sqrt{x}(x+1)} = \\ &= x \arctg \sqrt{x} - \frac{1}{2} \int \frac{\sqrt{x} dx}{1+x} \\ \int \frac{\sqrt{x} dx}{1+x} \text{ integralda } \sqrt{x} &= t \text{ desak, } x = t^2, \quad dx = 2t dt \text{ bo'lib} \end{aligned}$$

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x} dx &= \int \frac{t \cdot 2t}{1+t^2} dt = 2 \int \frac{t^2 dt}{1+t^2} = 2 \int \frac{1+t^2-1}{1+t^2} dt = \\ &= 2 \int \left(\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2} \right) dt = 2 \left[\int dt - \int \frac{dt}{1+t^2} \right] = \\ &= 2t - \arctg t + c = 2\sqrt{x} - 2\arctg \sqrt{x} + c \end{aligned}$$

Bularga ko'ra berilgan integral quyidagiga teng bo'ladi.

$$I = \int \arctg \sqrt{x} dx = x \arctg \sqrt{x} - \sqrt{x} + \arctg \sqrt{x} + c = (x+1) \arctg \sqrt{x} - \sqrt{x} + c$$

3. Ratsional kasrlarni integrallash

Sodda integrallarni hisoblash qoidalari.

$$1. I = \int \frac{dx}{ax+b} = \frac{1}{a} \int \frac{d(ax+b)}{ax+b} = \frac{1}{a} \ln|ax+b| + c$$

$$dx = d(ax+b) = d(ax) + d(b) = a \frac{1}{a} dx$$

2. $I = \int \frac{Ax+B}{ax^2+bx+c} dx$ Bu ko'rinishdagi integrallarni hisoblash uchun biz to'la kvadrat ajratamiz.

$$\begin{aligned} I &= \int \frac{Ax+B}{ax^2+bx+c} dx = \frac{1}{a} \int \frac{Ax+B}{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}} = \\ &= \left[\begin{array}{l} x+\frac{b}{2a}=t, \quad x=t-\frac{b}{2a} \\ \frac{b^2-4ac}{4a^2}=k^2 \end{array} \right] = \frac{1}{a} \int \frac{At dt}{t^2-k^2} + \frac{1}{a} \int \frac{B-\frac{Ab}{2a}}{t^2-k^2} dt = \\ &= \frac{A}{a} \frac{1}{2} \int \frac{d(t^2-k^2)}{t^2-k^2} + \frac{B-\frac{Ab}{2a}}{a} \int \frac{dt}{t^2-k^2} = \\ &= \frac{A}{2a} \ln|t^2-k^2| + \frac{B-\frac{Ab}{2a}}{a} \frac{1}{2k} \ln\left|\frac{t-k}{t+k}\right| + c \end{aligned}$$

Irratsional ifodalarni integrallash.

$I = \int F(x, \sqrt[n]{ax+b}) dx$ ko'rinishdagi integrallar irratsional ifodali integral deyiladi. Bunday integrallarni hisoblash uchun $\sqrt[n]{ax+b} = t$ almashtirish bajaramiz.

$$ax+b = t^n, \quad x = \frac{1}{a}(t^n-b), \quad dx = \frac{1}{a} n t^{n-1} dt$$

Bularni berilgan integral ostidagi ifodaga qo'ysak, u quyidagi ko'rinishni oladi.

$$I = \int F(\sqrt[n]{ax+b}) dx = \int F\left[\frac{1}{a}(t^n-b), t\right] \frac{1}{a} n t^{n-1} dt$$

Integral ostidagi hosil bo'lgan ifodalar ratsional ifodalardir. Biz bunday integrallarni hisoblashni bilamiz.

4-Misol. $I = \int \frac{x dx}{\sqrt{x^2-1}}$ ni xisoblang.

$$I = \int \frac{x dx}{\sqrt{x^2-1}} = \left[\begin{array}{l} x^2-1=t^2 \\ x^2=t^2+1 \\ x=\sqrt{t^2+1} \end{array} \quad dx = \frac{2t dt}{2\sqrt{t^2+1}} \right] = \int \frac{\sqrt{t^2+1} t dt}{t\sqrt{t^2+1}} = \int dt = t + c = \sqrt{x^2-1} + c$$

2- usul.

$$I = \int \frac{x dx}{\sqrt{x^2 - 1}} = \frac{1}{2} (x^2 - 1)^{-\frac{1}{2}} d(x^2 - 1) = \frac{1}{2} \frac{(x^2 - 1)^{\frac{1}{2}}}{\frac{1}{2}} = \sqrt{x^2 - 1} + c$$

$$I = \int F(x, {}^\alpha \sqrt{ax+b}, {}^\beta \sqrt{ax+b}, \dots, {}^\gamma \sqrt{ax+b}) dx$$

Bu ko'rinishdagi integrallarni hisoblash uchun α, β, γ ko'rsatkichlarning eng kichik umumiy buluvchisini topamiz. Faraz qilaylik bular uchun eng kichik umumiy buluvchi n bo'lsin, u holda

$${}^n \sqrt{ax+b} = t, \quad ax+b = t^n, \quad x = \frac{1}{a}(t^n - b) \quad dx = \frac{1}{a}(nt^{n-1} dt)$$

$$I = \int F\left(\frac{1}{a}(t^n - b), (t^\alpha, t^\beta, \dots, t^\gamma) dt \cdot \frac{1}{a} \cdot n \cdot t^{n-1}\right)$$

4. Trigonometrik funksiyalar qatnashgan integrallarni hisoblash

$I = \int R(\sin x, \cos x) dx$ ko'rinishdagi integrallarni $\operatorname{tg} \frac{x}{2} = t$ almashtirish orqali ratsional funksiyalarning integrallariga keltiriladi.

Bizga ma'lumki,

$$\sin x = \frac{2 \operatorname{tg} \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{2t}{1+t^2};$$

$$\cos x = \frac{1 - \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2};$$

$$x = 2 \arctg t; dx = \frac{2dt}{1+t^2};$$

5- Misol: $I = \int \frac{dx}{3 + 5 \sin x + 3 \cos x}$ integral hisoblansin.

Yechish: Yuqoridagi belgilashlarga ko'ra integral ostidagi funksiya quyidagi ko'rinishni oladi.

$$\begin{aligned} I &= \int \frac{dx}{3 + 5 \sin x + 3 \cos x} = \int \frac{2dt}{(1+t^2)(3 + 5 \frac{2t}{1+t^2} + 3 \frac{1-t^2}{1+t^2})} = \\ &= 2 \int \frac{dt}{3 + 3t^2 + 10t + 3 - 3t^2} = \int \frac{dt}{5t + 3} = \frac{1}{5} \ln |5t + 3| = \frac{1}{5} \ln |5 \operatorname{tg} \frac{x}{2} + 3| + c; \end{aligned}$$

Ba'zi xususiy hollarda yuqoridagi integrallarni yechishda quyidagi ko'rinishdagi o'rniga qo'yishlardan foydalanamiz.

a) Agar $R(\sin x, \cos x)$ ifoda $\sin x$ ga nisbatan toq funksiya bo'lsa, ya'ni $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ u holda $\cos x = t$ o'rniga qo'yish bu funktsiyani ratsionallashtiriladi.

b) Agar $R(\sin x, \cos x)$ ifoda $\cos x$ ga nisbatan toq funksiya, ya'ni $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ bo'lsa, u holda integral $\sin x = t$ o'rniga qo'yish bilan ratsional funksiyalarni integrallashga keltiriladi.

v) Agar $R(\sin x, \cos x)$ ifoda $\sin x$ va $\cos x$ ga nisbatan juft funksiya, ya'ni $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ bo'lsa, u holda bu funksiya $\tan x = t$ o'rniga qo'yish bilan ratsionallashtiriladi. Bu holda

$$\cos^2 x = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + t^2} \quad \sin^2 x = \frac{\tan^2 x}{1 + \tan^2 x} = \frac{t^2}{1 + t^2} \quad x = \arctan t; dx = \frac{dt}{1 + t^2};$$

Agar $R(\tan x)$ bo'lsa, u holda integral ostidagi ifoda yana $\tan x = t$ o'rniga qo'yish bilan ratsionallashtiriladi.

6-Misol. Integralni toping.

$$\int \sin^3 x \cdot \cos^2 x dx$$

Yechish.

$$\begin{aligned} I &= \int \sin^3 x \cdot \cos^2 x dx = \int \sin^2 x \cdot \cos^2 x \cdot \sin x dx = \\ &= -\int \sin^2 x \cdot \cos^2 x d(\cos x) = -\int (1 - \cos^2 x) \cdot \cos^2 x d(\cos x) = \\ &= -\int (\cos^2 x - \cos^4 x) d(\cos x) = c - \frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x \end{aligned}$$

B) Agar $m > 0$ bo'lib toq bo'lsa, u holda $\sin x = t$, $\cos x dx = dt$ o'rniga qo'yish orqali ratsionallashtiriladi.

7-Misol. $I = \int \sin^4 x dx$ integralni yeching.

Yechish.

$$\begin{aligned} I &= \int \sin^4 x dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx = \\ &= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx = \frac{1}{4} (x - \sin 2x + \int \cos^2 2x dx) = \\ &= \frac{1}{4} (x - \sin 2x + \frac{1}{2} \int (1 + \cos 4x) dx) = \frac{1}{4} (x - \sin 2x + \frac{1}{2} x + \frac{1}{8} \sin 4x) + c = \frac{1}{4} \left(\frac{3x}{2} - \sin 2x + \frac{1}{8} \sin 4x \right) + c; \end{aligned}$$

G) Agar $m, n \leq 0$ bo'lib ulardan biri toq bo'lsa, u holda surat va maxrajni $\sin x$ yoki $\cos x$ ga qaysiniki toq darajasiga qarab, qo'shimcha ko'paytirish usulidan foydalanamiz.

8-Misol. Integralni yeching.

$$\begin{aligned} I &= \int \frac{dx}{\sin x} \\ \int \frac{dx}{\sin x} &= \int \frac{\sin x dx}{\sin^2 x} = -\int \frac{d(\cos x)}{1 - \cos^2 x} = -\frac{1}{2} \ln \left| \frac{1 + \cos x}{1 - \cos x} \right| + c = \ln \sqrt{\tan^2 \frac{x}{2}} + c = \ln \left| \tan \frac{x}{2} \right| + c \end{aligned}$$

D) Agar $m + n < 0$ va juft bo'lsa, u holda $\tan x = t$ yoki $\cot x = t$ o'rniga qo'yishdan foydalanamiz.

$I = \int \sin mx \cdot \cos nx dx$, $I = \int \cos mx \cdot \cos nx dx$ va $I = \int \sin mx \cdot \sin nx dx$ ko'rinishdagi integrallar trigonometrik funksiyalar ko'paytmasini yig'indi shaklga keltirish orqali oson hisoblanadi.

9-Misol. $I = \int \sin 2x \cos 5x dx$ integralni xisoblang.

Yechish. $I = \int \sin 2x \cdot \cos 5x dx = \frac{1}{2} \int (\sin 7x + \sin(-3x)) dx =$
 $= \frac{1}{2} \int \sin 7x dx - \frac{1}{2} \int \sin 3x dx = -\frac{1}{14} \cos 7x + \frac{1}{6} \cos 3x + C.$

Misollar va masalalar

Quyidagi integrallarni hisoblang:

1. $\int \frac{dx}{(x+1) \sqrt[3]{\ln(x+1)}}.$

2. $\int \frac{\sin 3x}{\cos^2 3x} dx.$

3. $\int \frac{\sqrt{\operatorname{ctg} 7x}}{\sin^2 7x} dx.$

4. $\int \frac{\arcsin^2 5x}{\sqrt{1-25x^2}} dx.$

5. $\int e^{4-5x^2} x dx.$

6. $\int \frac{3x+2}{\sqrt{2x^2-1}} dx.$

7. $\int \frac{\sin 2x}{1+3\cos 2x} dx.$

8. $\int \frac{e^{3x} dx}{4-e^{6x}}.$

9. $\int \sin^2 (2x-1) dx.$

10. $\int \operatorname{tg}^4 \frac{2x}{3} dx.$

11. $\int \sin 3x \cos x dx.$

12. $\int \frac{dx}{4x^2-5x+4}.$